

# On Distribution of Zeros of Some Quazipolynoms

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## Abstract

In this paper we investigate distribution of zeros for only quasipolynom and obtain exactly lower-bound for their modulus.

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As is known [1, 2] in connection with the investigation of completeness of a system of eigen and adjoint elements of definite class of spectral problems that are not regular by Tamarkin-Rasulov [3, 4], there arises the necessity of studying properties of entire analytical functions of the form

$$f_k(\lambda) = e^\lambda + A_k \lambda^k \quad (1)$$

( $k$ -is natural, and  $A_k \neq 0$  - are complex constants), which is of special interest. In Ref. [5] studied zero sets of some classes of entire functions, gives a possibility to describe zeros of certain classes of entire functions of one and several variables in terms of growth of volume of these sets in certain polycylinders. In Ref. [6] studied the asymptotics of zeros for entire functions of the form  $\sin z + \int_{-1}^1 f(t)e^{izt}dt$  with  $f$  belonging to a space  $X \hookrightarrow L_1(-1, 1)$  possessing some minimal regularity properties. In Ref. [7] investigated a complete description of zero sets for some well-known subclasses of entire functions of exponential growth.

By taking these points into account, it may be argued that the investigation of distribution of zeros for quasipolynomial.

Note that in paper [8] we have obtained the function  $\Delta(\lambda)$  ( $\Delta(\lambda)$  -is called a characteristic function) which is an entire analytical function of a complex parameter  $\lambda$  and studying some its properties (for example, distribution of zeros, distance between two zeros, lower estimation for modules and etc.) is very important step in spectral theory of differential operators.

The present paper is continuation of [8] our work and the function of the form (1) is a special case of  $\Delta(\lambda)$ .

Introduce into consideration the following sets of points of the complex

surfare  $C$  :

$$\Omega_{R_1 R_2}(\lambda_0) = \{\lambda; R_1 \leq |\lambda - \lambda_0| \leq R_2\}, \quad \Omega_{R_1, R_2} = \Omega_{R_1, R_2}(0),$$

$$\Omega_R(\lambda_0) = \{\lambda; |\lambda - \lambda_0| \leq R\}, \quad \Omega_R = \Omega_R(0),$$

$$\Gamma_{kj}^S(h, R) = \{\lambda; Re\lambda + (-1)^S k \ln |\lambda|$$

$$= h, (-1)^j Jm\lambda < 0\} \cap \Omega_{R, \infty},$$

$$\Gamma_k^S(h, R) = \bigcup_{j=1}^2 \Gamma_{kj}^S,$$

$$\prod_{kj}^S(h, R) = \{\lambda; |Re\lambda + (-1)^S k \ln |\lambda|| \leq h, (-1)^j Jm\lambda < 0\} \cap \Omega_{R, \infty},$$

$$\prod_k^S(h, R) = \bigcup_{j=1}^2 \prod_{kj}^S(h, R). \quad (2)$$

$$T_{k1}^S(h, R) = \{\lambda; Re\lambda + (-1)^S k \ln |\lambda| < -h\} \cap \Omega_{R, \infty},$$

$$T_{k2}^S(h, R) = C \setminus T_{k1}^S(h, R) \cup \Pi_k^S(h, R) \cup \overline{\Omega}_{0, R},$$

$$T_k^S(h, R) = \bigcup_{j=1}^2 T_{kj}^S(h, R),$$

$$\Sigma_\delta^{(i)} = \left\{ \lambda; \left| \arg \lambda + (-1)^i \frac{\pi}{2} \right| < \delta \right\},$$

where  $0 \leq R_1 < R_2 < \infty$ ,  $R > 0$ ,  $h > 0$ ,  $i = 1, 2$ ;  $j = 1, 2$ ;  $S = 1, 2$ ;  $\delta > 0$ .

Now let's investigate some properties of the function  $f_k(\lambda)$ :

$$|f_k(\lambda)| \geq |A_k| |\lambda|^k \left[ 1 - |B_k| e^{Re\lambda - k \ln |\lambda|} \right] \geq$$

$$|A_k| |\lambda|^k \left[ 1 - |B_k| e^{-h} \right],$$

where  $|B_k| = \frac{1}{|A_k|}$ .

In the case  $\lambda \in T_{k1}^1(h, R)$  and choosing  $h > \ln 2 |B_k|$  we arrive at the estimation of the form

$$|f_k(\lambda)| \geq \frac{1}{2} |A_k| |\lambda|^k, \quad (3)$$

and for  $\lambda \in T_{k2}^2(h, R)$  we arrive at the following estimation of the form

$$|f_k(\lambda)| \geq \frac{1}{2} |e^\lambda|, \text{ if } h > \ln 2 |A_k| \quad (4)$$

Thus it was proved:

Lemma 1. The function  $f_k(\lambda)$  at the sufficiently large  $h > 0$ ,  $R > 0$  in the domain  $T_{k1}^1(h, R)$  and  $T_{k2}^2(h, R)$  has not zeros. And the estimations (3) and (4) are true for it. Absence of zeros of the function  $f_k(\lambda)$  in the domain  $T_{k1}^1(h, R)$  and  $T_{k2}^2(h, R)$  is obvious from the estimations (3) and (4). Proceeding from the definition of the curvilinear bands  $\Pi_{kj}^S(h, R)$  the following is easily proved.

Lemma 2. For any  $\delta > 0$  and  $h > 0$  we can find  $R > 0$ , such that

$$\prod_{kj}^S(h, R) \subset \Sigma_\delta^{(1)} \cup \Sigma_\delta^{(2)}$$

Let's prove now the following lemma.

Lemma 3. The function  $f_k(\lambda)$  in the complex surface  $C$  has denumerable sets of zeros  $\{\lambda_{\nu k}\}$  with unique limit point  $\lambda = \infty$  which at sufficiently large  $h > 0$ ,  $R > 0$ , is situated in the domain  $\Pi_k^1(h, R) \cup \overline{\Omega}_{0,R}$ . These zeros allow the asymptotic representation:

$$\lambda_{\nu k} = \ln \frac{|A_k|}{[2\pi |\nu|]^k} + i \left( 2\pi\nu + \pi + \frac{\pi k}{2} + \arg A_k \right) + o \left( \frac{\ln |\nu|}{\nu} \right) \quad (5)$$

Proof. The assertion of the first part of the lemma follows from the general theory of Picard [9] and from the lemma 1. Prove the second part of the lemma.

$$f_k(\lambda) = 0, \quad e^\lambda + A_k \lambda^k = 0,$$

$$e^\lambda \cdot \lambda^{-k} = -A_k, \quad e^\lambda \cdot e^{-k \ln \lambda} = -A_k,$$

$$\lambda - k \ln |\lambda| = \ln |A_k| + i (\arg(-A_k) + 2\pi\nu).$$

Make the substitution  $\lambda - 2\pi\nu i = \xi_\nu$ , then

$$\begin{aligned} \xi_\nu &= \ln |A_k| + i \arg(-A_k) + k \ln \lambda = \\ &= \ln |A_k| + i (\arg A_k + \pi) + k \ln (2\pi\nu i + \xi_\nu) = \\ &= \ln |A_k| + i (\arg A_k + \pi) + k \ln \left[ 2\pi\nu i \left( 1 + \frac{\xi_\nu}{2\pi\nu i} \right) \right] = \\ &= \ln |A_k| + i (\arg A_k + \pi) + k \ln(2\pi\nu i) + k \ln \left( 1 + \frac{\xi_\nu}{2\pi\nu i} \right). \end{aligned}$$

Since

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \ln \left( 1 + \frac{\xi_\nu}{2\pi\nu i} \right) &= 0, \quad \ln \left( 1 + \frac{\xi_\nu}{2\pi\nu i} \right) = \\ &= 0 \left( \frac{\xi_\nu}{2\pi\nu i} \right) = 0 \left( \frac{\ln |\nu|}{\nu} \right). \end{aligned}$$

$$\xi_\nu = \ln |A_k| + i (\arg A_k + \pi) + k \ln(2\pi\nu i) + O \left( \frac{\ln |\nu|}{\nu} \right),$$

then we find

$$\lambda_\nu = \ln \frac{|A_k|}{[2\pi |\nu|]^{-k}} + i \left( 2\pi\nu + \pi + \frac{k\pi}{2} + \arg A_k \right) + O \left( \frac{\ln |\nu|}{\nu} \right).$$

From (5) we see that at large  $|\nu|$  we have

$$|\lambda_{\nu+1} - \lambda_\nu| = 2\pi + o(1) \tag{6}$$

Consequently there exists  $\delta > 0$  such that the circles of  $\Omega_\delta(\lambda_\nu)$  are mutually exclusive and at the sufficiently large  $h > 0, R > 0$  wholly lie in the domain

$\prod_k^1 (h, R) \cup \overline{\Omega}_{0,R}$ . From the asymptotic formulae (5) and (6) we see that straight lines

$$l_{\nu,k} = \left\{ \lambda : Jm\lambda = Jm\lambda_\nu - \left( \pi + \frac{\pi k}{2} + \arg A_k \right) \right\}$$

are perpendicular to the imaginary axis  $Re\lambda = 0$  at all possible different, sufficiently large (by the modulus) values of  $\nu$ , are different. These lines divide the domains  $\prod_k^1 (h, R)$  into the curvilinear quadrangles  $D_{\nu k} = D_{\nu k}(h, R)$  with lateral boundaries on the lines  $\gamma_k(-h, R)$ ,  $\gamma_k(h, R)$  and lies on the straight lines  $l_{\nu-1,k} > l_{\nu,k}$ . The length of the diagonal of a quadrangle denote by

$$d_\nu = \sup_{\lambda, \mu \in D_{\nu,k}} |\lambda - \mu|.$$

Introduce the following notation

$$\prod_k^1 (h, R, \delta) = \prod_k^1 (h, R) \setminus \bigcup_\nu \Omega_{0,\delta}(\lambda_\nu),$$

where the sign of unification is propagated on all  $\nu$  such that

$$\lambda_\nu \in \prod_k^1 (h, R),$$

$$D_{k\nu}^\delta = D_{k\nu} \setminus \Omega_{0,\delta}(\lambda_\nu).$$

The following lemma is true.

Lemma 4. There exists the constant  $\delta > 0$ , such that at  $\lambda \in \prod_k^1 (h, R, \delta)$  it holds the inequality

$$|f_k(\lambda)| \geq C_\delta |\lambda|^k. \quad (7)$$

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